

## 2023 Mathematics *Standards of Learning*

### Understanding the Standards – Algebra, Functions, and Data Analysis

The Understanding the Standards document includes the mathematics understandings and key concepts that assist teachers in planning standards-focused instruction of the 2023 Mathematics *Standards of Learning* for Algebra, Functions, and Data Analysis. The Understanding the Standards document includes definitions, explanations, and examples regarding each mathematics standard and describes what students should know (core knowledge) as a result of the instruction specifically aligned to the course/grade level.

#### Algebra and Functions

**AFDA.AF.1 The student will investigate, analyze, and compare linear, quadratic, and exponential function families, algebraically and graphically, using transformations.**

*Students will demonstrate the following Knowledge and Skills:*

- a) Identify graphs and equations of parent functions for linear, quadratic, and exponential function families.
- b) Describe the transformation from the parent function given the equation or the graph of the function.
- c) Determine and analyze whether a linear, quadratic, or exponential function best models a given representation, including those in context.
- d) Write the equation of a linear, quadratic, or exponential function, given a graph, using transformations of the parent function.
- e) Use a graphical or algebraic representation of a function to solve problems within a context, graphically and algebraically, when appropriate.
- f) Graph a function given the equation of a function, using transformations of the parent function. Use technology to verify transformations of functions.
- g) Compare and contrast linear, quadratic, and exponential functions using multiple representations (e.g., graphs, tables, equations, verbal descriptions).

**AFDA.AF.1 The student will investigate, analyze, and compare linear, quadratic, and exponential function families, algebraically and graphically, using transformations.**

*Additional Content Background and Instructional Guidance:*

- Connections between and among multiple representations (graphs, tables, equations, with and without context) of a function can be made.
- The graphs/equations for a family of functions can be determined using a transformational approach.
- Parent functions are the most basic form of a function. The parent function exists prior to any transformations occurring. Some of the more commonly used parent functions include:
  - Linear:  $f(x) = x$
  - Quadratic:  $g(x) = x^2$
  - Exponential:  $h(x) = a^x$

**AFDA.AF.1 The student will investigate, analyze, and compare linear, quadratic, and exponential function families, algebraically and graphically, using transformations.**

*Additional Content Background and Instructional Guidance:*

- Function families consist of a parent function and all transformations of the parent function.
- The graph of a parent function serves as an anchor graph from which other graphs are derived using transformations.
- The transformation of a function, called a pre-image, changes the size, shape, and/or position of the function to a new function, called the image.
- Transformations of graphs include:
  - Translations (horizontal and/or vertical shifting of a graph) which is represented by the function notation  $f(x) + k$  and  $f(x + k)$ ;
  - Reflections over the y-axis which is represented by the function notation  $f(-x)$ ;
  - Reflections over the x-axis which is represented by the function notation  $-f(x)$ ; and,
  - Dilations (stretching and compressing of graphs).
    - If a graph is dilated parallel to the x-axis, all x-values are modified by the same scale factor, which is represented by the function notation  $f(kx)$ ; where:
      - $|k| > 1$  is a horizontal compression.
      - $0 < |k| < 1$  is a horizontal stretch.
    - If a graph is dilated parallel to the y-axis, all y-values are modified by the same scale factor, which is represented by the function notation  $kf(x)$ ; where:
      - $|k| > 1$  is a vertical stretch.
      - $0 < |k| < 1$  is a vertical compression.
- Knowledge of transformational graphing using parent functions can be used to verify a mathematical model from a scatterplot that approximates the data.
- The equation of a function can be determined by determining the transformations of the graph of the parent function.
- The most appropriate representation of a function depends on the questions to be answered and/or the analysis to be done.
- The graph of a function can be determined by identifying the transformations from the equation of a function and applying them to the graph of the parent function.
- Contextual data may best be represented as a table, a graph, or a formula.
- Given data may be represented as discrete points or as a continuous graph with respect to the context.
- The equation of a linear function can be determined by two points on the line or by the slope and a point on the line.

**AFDA.AF.2 The student will investigate and analyze characteristics of the graphs of linear, quadratic, exponential, and piecewise-defined functions.**

*Students will demonstrate the following Knowledge and Skills:*

- a) Determine the domain and range of a function given a graphical representation, including those limited by contexts.
- b) Identify intervals on a graph for which a function is increasing, decreasing, or constant.
- c) Given a graph, identify the location and value of the absolute maximum and absolute minimum of a function over the domain of a function.
- d) Given a graph, determine the zeros and intercepts of a function.
- e) Describe and recognize the connection between points on the graph and the value of a function.
- f) Describe the end behavior of a function given its graph.
- g) Identify horizontal and/or vertical asymptotes from the graph of a function if they exist.
- h) Describe and relate the characteristics of the graphs of linear, quadratic, exponential, and piecewise-defined functions, including those in contextual situations.

**AFDA.AF.2 The student will investigate and analyze characteristics of the graphs of linear, quadratic, exponential, and piecewise-defined functions.**

*Additional Content Background and Instructional Guidance:*

- In AFDA, the characteristics of functions are investigated and analyzed from only a graph. Investigation and analysis from equations occurs in Algebra 2 as part of the learning progression.
- Functions are used to model contextual situations.
- Functions describe the relationship between two variables where each input is paired to a unique output.
- A relation is a function if and only if each element in the domain is paired with a unique element of the range.
- The domain of a function is the set of all possible values of the independent variable.
- The range of a function is the set of all possible values of the dependent variable.
- For each  $x$  in the domain of  $f$ ,  $x$  is a member of the input of the function  $f$ ,  $f(x)$  is a member of the output of  $f$ , and the ordered pair  $(x, f(x))$  is a member of  $f$ .
- The domain of a function may be restricted algebraically, graphically, or by the contextual situation modeled by the function.
- A value  $x$  in the domain of  $f$  is an  $x$ -intercept or a zero of a function  $f$  if and only if  $f(x) = 0$ .
- The  $x$ -intercept is the point at which the graph of a relation or function intersects with the  $x$ -axis. It can be expressed as a value or a coordinate.
- The  $y$ -intercept is the point at which the graph of a relation or function intersects with the  $y$ -axis. It can be expressed as a value or a coordinate.
- Unless specific information is provided on the graph or the equation of the function is given, the characteristics of a function, such as roots, from a graph are estimations.

**AFDA.AF.2 The student will investigate and analyze characteristics of the graphs of linear, quadratic, exponential, and piecewise-defined functions.***Additional Content Background and Instructional Guidance:*

- Given a polynomial function  $f(x)$ , the following statements are equivalent for any real number,  $k$ , such that  $f(k) = 0$ :
  - $k$  is a zero of the polynomial function  $f(x)$  located at  $(k, 0)$ ;
  - $k$  is a solution or root of the polynomial equation  $f(x) = 0$ ;
  - the point  $(k, 0)$  is an  $x$ -intercept for the graph of polynomial  $f(x) = 0$ ; and
  - $k$  is an  $x$ -intercept for the graph of the polynomial; and
  - $(x - k)$  is a factor of polynomial  $f(x)$ .
- A function can be described on an interval as increasing, decreasing, or constant over a specified interval or over the entire domain of the function.
- A function,  $f(x)$ , is increasing over an interval if the graph of  $f(x)$  consistently increases over the interval as the  $x$  values increase.
- A function,  $f(x)$ , is decreasing over an interval if the graph of  $f(x)$  consistently decreases over the interval as the  $x$  values increase.
- A function,  $f(x)$ , is constant over an interval if the graph of  $f(x)$  remains constant over the interval as the  $x$  values increase.
- Exponential functions are either strictly increasing or strictly decreasing.
- A turning point is a point on a continuous interval where the graph changes from increasing to decreasing or from decreasing to increasing.
- The absolute maximum of a function is a point on a graph of a function where the function's value is at its highest point across the entire domain of the function.
- The absolute minimum of a function is a point on a graph of a function where the function's value is at its lowest point across the entire domain of the function.
- A function is continuous on an interval if the function is defined for every value in the interval and there are no breaks in the graph. A continuous function can be drawn without lifting the pencil.
- Continuous and discontinuous functions can be identified by their equations or graphs.
- Asymptotes can be used to describe local behavior and end behavior of graphs. They are lines or other curves that approximate the graphical behavior of a function.
- End behavior describes a function's values as  $x$  approaches positive or negative infinity.

**AFDA.AF.2 The student will investigate and analyze characteristics of the graphs of linear, quadratic, exponential, and piecewise-defined functions.***Additional Content Background and Instructional Guidance:*

- Solutions and intervals may be expressed in different formats, including set notation, using equations and inequalities, or interval notation. Examples may include:

Equation/ Inequality	Set Notation	Interval Notation
$x = 3$	$\{3\}$	
$x = 3$ or $x = 5$	$\{3, 5\}$	
$0 \leq x < 3$	$\{x \mid 0 \leq x < 3\}$	$[0, 3)$
$y \geq 3$	$\{y \mid y \geq 3\}$	$[3, \infty)$
Empty (null) set $\emptyset$	$\{ \}$	

**AFDA.AF.3 The student will represent and interpret contextual situations with constraints that require optimization using linear programming techniques, including systems of linear equations or inequalities, solving graphically and when appropriate, algebraically.**

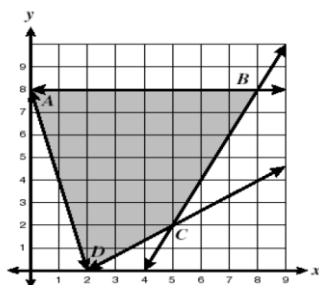
*Students will demonstrate the following Knowledge and Skills:*

- Represent and interpret contextual problems requiring optimization with systems of linear equations or inequalities.
- Solve systems of no more than four equations or inequalities graphically and when appropriate, algebraically.
- Identify the feasible region of a system of linear inequalities.
- Identify the coordinates of the vertices of a feasible region.
- Determine and describe the maximum or minimum value for the function defined over a feasible region.
- Interpret the validity of possible solution(s) algebraically, graphically, using technology, and in context and justify the reasonableness of the answer(s) or the solution method in context.

**AFDA.AF.3 The student will represent and interpret contextual situations with constraints that require optimization using linear programming techniques, including systems of linear equations or inequalities, solving graphically and when appropriate, algebraically.**

*Additional Content Background and Instructional Guidance:*

- Linear programming models an optimization process.
- A linear programming model consists of a system of constraints and an objective quantity that can be maximized or minimized.
- Any maximum or minimum value will occur at a corner point of a feasible region.
- The feasible region of a linear programming problem is convex, and the maximum or minimum quantity is determined at one of the vertices (corner points) of this region:
  - Define the variables.
  - Give the equation for the quantity to be maximized or minimized.
  - Set up a system of linear inequalities for the constraints.
  - Graph the feasibility region.
  - Give the coordinates of all corner points.
  - Evaluate the quantity at each corner point.
  - Determine the solution and state your results in a complete sentence.
- The graph below of the linear programming model consists of polygon  $ABCD$  and its interior.



## Data Analysis

**AFDA.DA.1 The student will apply the data cycle (formulate questions; collect or acquire data; organize and represent data; and analyze data and communicate results) with a focus on representing bivariate data in scatterplots and determining the curve of best fit using linear, quadratic, and exponential functions.**

*Students will demonstrate the following Knowledge and Skills:*

- Formulate investigative questions that require the collection or acquisition of bivariate data, where exactly two of the variables are quantitative.
- Collect or acquire bivariate data from a representative sample to answer an investigative question.
- Represent bivariate data with a scatterplot using technology and describe how the variables are related in terms of the given context.
- Make predictions, decisions, and critical judgments using data, scatterplots, or the equation(s) of the mathematical model.

**AFDA.DA.1 The student will apply the data cycle (formulate questions; collect or acquire data; organize and represent data; and analyze data and communicate results) with a focus on representing bivariate data in scatterplots and determining the curve of best fit using linear, quadratic, and exponential functions.**

*Additional Content Background and Instructional Guidance:*

- In AFDA, this standard focuses on individual functions. Algebra 2 builds on this concept by combining functions within data sets.
- Data and scatterplots may indicate patterns that can be represented with a function.
- Categorical variables can be added to a scatterplot using color or different symbols.
- Technology such as spreadsheets and graphing utilities can be used to collect, organize, represent, and generate a mathematical model for a set of data.
- Knowledge of transformational graphing using parent functions can be used to verify a mathematical model from a scatterplot that approximates the data.
- Data that fit linear ( $y = mx + b$ ), quadratic ( $y = ax^2 + bx + c$ ), or exponential ( $y = ab^x$ ) models arise from contextual situations.
- The mathematical model of the relationship among a set of data points can be used to make predictions where appropriate.
- An equation which represents the curve of best fit for the relationship among a set of data points can be used to make predictions where appropriate.
- Rounding that occurs during intermediate steps of problem solving may reduce the accuracy of the final answer.
- Two variables may be strongly associated without a cause-and-effect relationship existing between them.
- Evaluation of the reasonableness of a mathematical model of a contextual situation involves asking questions including:

**AFDA.DA.1 The student will apply the data cycle (formulate questions; collect or acquire data; organize and represent data; and analyze data and communicate results) with a focus on representing bivariate data in scatterplots and determining the curve of best fit using linear, quadratic, and exponential functions.**

*Additional Content Background and Instructional Guidance:*

- “Is there another curve (linear, quadratic, or exponential) that better fits the data?”
- “Does the curve of best fit make sense?”
- “Could the curve of best fit be used to make reasonable predictions?”
- Each data point may be considered to be comprised of two parts: fit (the part explained by the model) and residual (the result of chance variation or of variables not measured).
- $\text{Residual} = \text{Actual} - \text{Fitted}$
- Least squares regression generates the equation of the line that minimizes the sum of the squared distances between the data points and the line.
- A correlation coefficient measures the degree of association between two variables that are related linearly.



**AFDA.DA.2 The student will apply the data cycle (formulate questions; collect or acquire data; organize and represent data; and analyze data and communicate results) with a focus on the design and implementation of an experiment and/or observational study.**

*Students will demonstrate the following Knowledge and Skills:*

- a) Formulate questions that can be addressed with data and assess the type of data relevant to the question (e.g., quantitative versus categorical).
- b) Investigate, describe, and determine best sampling techniques, such as simple random sampling, stratified sampling, and cluster sampling.
- c) Plan and conduct an experiment and/or observational study. The experimental design should address control, randomization, and minimization of experimental error.
- d) Collect or acquire data to answer a statistical question.
- e) Recognize that data may contain errors, have missing values, or may be biased, and make decisions about how to account for these issues.
- f) Identify biased sampling methods.
- g) Given a plan for an observational study, identify possible sources of bias, and describe ways to reduce bias.
- h) Select, create, and use appropriate visual representations of data to brainstorm solutions.
- i) Use appropriate statistical methods to analyze data.
- j) Communicate the description of an experiment and/or observational study, the resulting data, analysis, and the validity of the conclusions.

**AFDA.DA.2 The student will apply the data cycle (formulate questions; collect or acquire data; organize and represent data; and analyze data and communicate results) with a focus on the design and implementation of an experiment and/or observational study.**

*Additional Content Background and Instructional Guidance:*

- Types of Data:
  - Quantitative data is represented numerically, including anything that can be counted, measured, or given a numerical value.
  - Qualitative/categorical data is information that cannot be counted, measured, or easily expressed using numbers.
- The purpose of sampling is to provide sufficient information so that population characteristics may be inferred.
- Sampling Techniques:
  - A simple random sample is a subset of a statistical population in which each member of the subset has an equal probability of being chosen.
  - Stratified random sampling is the process of creating subgroups in a dataset according to various factors, such as age, gender, income level, or education.
  - Cluster sampling is used when natural groups are present in a population. The whole population is subdivided into clusters, or groups, and random samples are then collected from each group.

**AFDA.DA.2 The student will apply the data cycle (formulate questions; collect or acquire data; organize and represent data; and analyze data and communicate results) with a focus on the design and implementation of an experiment and/or observational study.**

*Additional Content Background and Instructional Guidance:*

- Principles of experimental design include –
  - A control group is a group in the experiment which a variable is not being tested, such as a test subject that does not receive any treatment. Control groups serve as important benchmarks to compare the results of the experimental group, or the group that is being experimented on.
  - Randomization is the process of assigning participants to treatment and control groups, assuming that each participant has an equal chance of being assigned to any group.
  - It is impossible to eliminate experimental error; however, random error can be reduced by taking repeated measurements, using a large sample, and controlling extraneous variables. Systematic error can be avoided through the careful design of sampling, data collection, and analysis procedures.
- Data collection is the process of gathering, measuring, and analyzing accurate data.
- Data limitations occur in different ways, such as data quality, sample size, assumptions, methods, or interpretation. These limitations can affect the validity, reliability, and generalizability of the results.
- Data may contain errors, have missing values, or be biased.
- Poor data collection can lead to misleading and meaningless conclusions.
- Data can be presented in various forms (e.g., written summaries, visual representations: graphs, charts, tables, maps, etc.)
- The value of a sample statistic may vary from sample to sample, even if the simple random samples are taken repeatedly from the population of interest.
- Experiments must be carefully designed to detect a cause-and-effect relationship between variables.
- The precision, accuracy, and reliability of data collection can be analyzed and described.

**AFDA.DA.3 The student will calculate and interpret probabilities, including those in contextual situations.**

*Students will demonstrate the following Knowledge and Skills:*

- Analyze, interpret, and make predictions based on theoretical probability.
- Calculate conditional probabilities for dependent, independent, and mutually exclusive events.
- Represent and calculate probabilities using Venn diagrams, probability trees, organized lists, two-way tables, simulations, or other probability models.
- Interpret probabilities from simulations or experiments to make informed decisions and justify the rationale.
- Define and give contextual examples of complementary, dependent, independent, and mutually exclusive events.
- Given two or more events in a problem setting, determine whether the events are complementary, dependent, independent, and/or mutually exclusive.
- Compare and contrast permutations and combinations, including those in contextual situations.
- Calculate the number of permutations of  $n$  objects taken  $r$  at a time, without repetition.
- Calculate the number of combinations of  $n$  objects taken  $r$  at a time, without repetition.

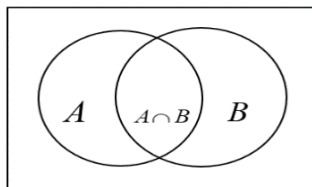
**AFDA.DA.3 The student will calculate and interpret probabilities, including those in contextual situations.**

*Additional Content Background and Instructional Guidance:*

- Theoretical probability in mathematics refers to the likelihood of an event occurring based on pure mathematics. The formula for calculating the theoretical probability of an event  $A$  is:

$$P(A) = \frac{\text{number of desired outcomes}}{\text{total number of possible outcomes}}$$

- A sample space is the set of all possible outcomes of a random experiment.
- An event is a subset of the sample space.
- $P(E)$  is a way to represent the probability that the event  $E$  occurs.
- Mutually exclusive events are events that cannot both occur simultaneously.
- If  $A$  and  $B$  are mutually exclusive, then  $P(A \cup B) = P(A) + P(B)$ .
- The complement of event  $A$  consists of all outcomes in which event  $A$  does not occur.
- $P(B|A)$  is the probability that  $B$  will occur given that  $A$  has already occurred.  $P(B|A)$  is called the conditional probability of  $B$  given  $A$ .
- Venn diagrams may be used to examine conditional probabilities.



$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A \cap B) = P(A)P(B|A)$$

**AFDA.DA.3 The student will calculate and interpret probabilities, including those in contextual situations.**

*Additional Content Background and Instructional Guidance:*

- Two events,  $A$  and  $B$ , are independent if the occurrence of one does not affect the probability of the occurrence of the other. If  $A$  and  $B$  are not independent, then they are said to be dependent.
- If  $A$  and  $B$  are independent events, then  $P(A \cap B) = P(A)P(B)$ .
- The Law of Large Numbers states that as a procedure is repeated again and again, the relative frequency probability of an event tends to approach the actual probability.
- The Fundamental Counting Principle states that if one decision can be made  $n$  ways and another can be made  $m$  ways, then the two decisions can be made  $nm$  ways.
- A permutation is the number of possible ways to arrange a group of objects without repetition and when order matters (e.g., the outcome 1, 2, 3 is different from the outcome 3, 2, 1 when order matters; therefore, both arrangements would be included in the possible outcomes).
- A combination is the number of possible ways to select or arrange objects when there is no repetition and order does not matter (e.g., the outcome 1, 2, 3 is the same as the outcome 3, 2, 1 when order does not matter; therefore, both arrangements would not be included in the possible outcomes).

**AFDA.DA.4 The student will describe and apply the properties of normal distribution, including those in contextual situations.**

*Students will demonstrate the following Knowledge and Skills:*

- a) Identify and describe the properties of a normal distribution.
- b) Determine when the normal distribution is a reasonable representation of the data.
- c) Describe how the mean and the standard deviation affect the graph of the normal distribution.
- d) Calculate and interpret the z-score for a data point, given the mean and the standard deviation.
- e) Compare two sets of normally distributed data using a standard normal distribution and z-scores, given the mean and the standard deviation.
- f) Represent probability as the area under the curve of a standard normal distribution.
- g) Determine probabilities associated with areas under the standard normal curve, using technology or a table of Standard Normal Probabilities.
- h) Investigate, represent, and determine relationships between a normally distributed data set and its descriptive statistics.

**AFDA.DA.4 The student will describe and apply the properties of normal distribution, including those in contextual situations.**

*Additional Content Background and Instructional Guidance:*

- A normal distribution curve is the family of symmetrical, bell-shaped curves defined by the mean and the standard deviation of a data set. The arithmetic mean ( $\mu$ ) is located on the line of symmetry of the curve and is approximately equivalent to the median and mode of the data set.
- There are data sets that cannot be represented by a smooth or normal curve.
- A very large data set provides a representation that can closely approximate the population.
- Summary statistics include measures of center (mean, median, mode) and dispersion or spread (variance and standard deviation). These statistics can be used to approximate the shape of the distribution.
- Analysis of the descriptive statistical information generated by a univariate data set includes the relationships between central tendency, dispersion, and position.
- Descriptive statistics include measures of center (mean, median, mode) and dispersion or spread (variance and standard deviation).
- Variance ( $\sigma^2$ ) and standard deviation ( $\sigma$ ) measure the spread of data about the mean in a data set.
- Standard deviation is expressed in the original units of measurement of the data.
- The greater the value of the standard deviation, the further the data tends to be dispersed from the mean.
- To develop an understanding of standard deviation as a measure of dispersion (spread), students should have experience analyzing the formulas for and the relationship between

**AFDA.DA.4 The student will describe and apply the properties of normal distribution, including those in contextual situations.**

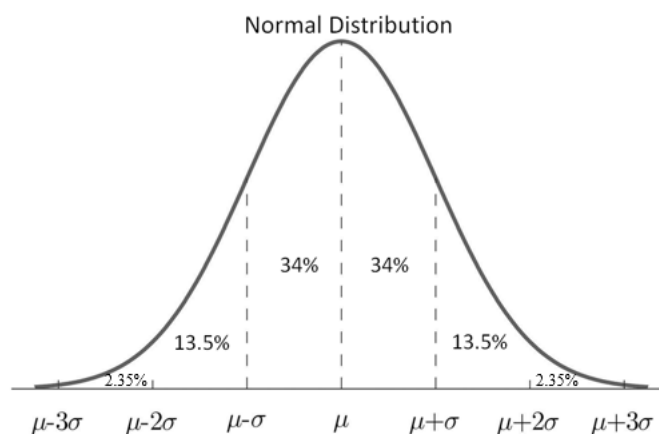
*Additional Content Background and Instructional Guidance:*

variance and standard deviation. The formulas for standard deviation and variance have been provided; however, students do not have to calculate these values.

$$\text{standard deviation: } \sigma = \sqrt{\frac{\sum_{i=1}^n (x_i - \mu)^2}{n}}$$

$$\text{variance } (\sigma^2) = \frac{\sum_{i=1}^n (x_i - \mu)^2}{n}$$

- The normal distribution curve is a family of symmetrical curves defined by the mean and the standard deviation.
- The mean and standard deviation of a normal distribution affect the location and shape of the curve. The vertical line of symmetry of the normal distribution falls at the mean. The greater the standard deviation, the wider (“flatter” or “less peaked”) the distribution of the data.
- Areas under the curve represent probabilities associated with continuous distributions.
- The normal curve is a probability distribution and the total area under the curve is 1.
- For a normal distribution, approximately 68 percent of the data fall within one standard deviation of the mean, approximately 95 percent of the data fall within two standard deviations of the mean, and approximately 99.7 percent of the data fall within three standard deviations of the mean. This is often referred to as the Empirical Rule or the 68 – 95 – 99.7 rule.



NOTE: This chart illustrates percentages that correspond to subdivisions in one standard deviation increments. Percentages for other subdivisions require the table of Standard Normal Probabilities or a graphing utility.

- A standard normal distribution is the set of all z-scores.
- The mean of the data in a standard normal density function is 0 and the standard deviation is 1. This allows for the comparison of unlike data.

**AFDA.DA.4 The student will describe and apply the properties of normal distribution, including those in contextual situations.**

*Additional Content Background and Instructional Guidance:*

- A  $z$ -score is a derived score from a given normal distribution.

$$\text{z-score: } z = \frac{x - \mu}{\sigma}$$

- A  $z$ -score derived from a particular data value tells how many standard deviations that data value falls above or below the mean of the data set. The  $z$ -score is positive if the data value lies above the mean and negative if the data value lies below the mean.
- A  $z$ -score is a measure of position derived from the mean and standard deviation of data. The amount of data that falls within 1, 2, or 3 standard deviations of the mean is constant and the basis of  $z$ -score data normalization.